HOW TURBULENCE ORGANIZES ITSELF
A STORY IN TWO DIMENSIONS

Anna Frishman, Technion
Navier-Stokes equation:

\[ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{f} ; \quad \nabla \cdot \mathbf{v} = 0 \]
Navier-Stokes equation:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{f} ; \quad \nabla \cdot \mathbf{v} = 0$$

High Reynolds number: $\text{Re} = \frac{UL}{\nu} \gg 1$, $U$ – typical velocity, $L$ – typical scale
TURBULENCE

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High Reynolds number: \( \text{Re} = \frac{UL}{\nu} \gg 1 \), \( U \) – typical velocity, \( L \) – typical scale

The flow is chaotic – the velocity is a random field
TURBULENCE

Navier- Stokes equation:

\[ \partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \nu \nabla^2 \vec{v} + f; \quad \nabla \cdot \vec{v} = 0 \]

The flow is chaotic – the velocity is a random field

Many excited degrees of freedom

**Strongly interacting** (non-linear), **Non local** (pressure),

**Out of equilibrium** (broken detailed balance)
FULLY DEVELOPED TURBULENCE

3d:

Energy injection

Viscous dissipation
THE INERTIAL RANGE

3d:

IR cutoff: forcing scale
UV cutoff: viscous scale
DISSIPATIVE ANOMALY

\[ \epsilon = \nu \left( \nabla \nu(0) \right)^2 \]

\[ D = \epsilon L / U^3 \]

Ishihara, Gotoh, Kaneda (2009)
$\lim_{\nu \to 0} \lim_{r \to 0} \nu \langle \nabla v(r) \nabla v(0) \rangle \to \lim_{\nu \to 0} \nu \left( \nabla v(0) \right)^2 \to \epsilon$

$(\lim_{r \to 0} \lim_{\nu \to 0} \nu \langle \nabla v(r) \nabla v(0) \rangle \to 0)$
The scaling of velocity structure functions:

\[ \langle u_l^n \rangle \propto r^{\zeta_n} \quad u_l = \vec{u} \cdot \vec{r} / r \]

\[ \vec{r} \] - The separation between two points.

\[ \vec{u} \] - The velocity difference between the points,
The scaling of velocity structure functions:

\[ \langle u_l^n \rangle \propto r^{\zeta_n} \quad u_l = \tilde{u} \cdot \tilde{r} / r \]

\[ \tilde{\zeta}_n - \text{are universal} \]
The scaling of velocity structure functions:

\[ \langle u_l^n \rangle \propto r^{\zeta_n} \quad u_l = \bar{u} \cdot \hat{r}/r \]

\[ \zeta_n - \text{are universal} \]

Dimensional analysis:

\[ u_l \sim (\epsilon r)^{\frac{1}{3}} \quad \zeta_n = n/3 \]
In the inertial range of scales:

\[ \nabla^i \langle u^i u^2 \rangle = -4\epsilon. \]

\[ \lim_{\nu \to 0} \nu \langle (\nabla v)^2 \rangle = \epsilon \]
In the inertial range of scales: \( \nabla^i \langle u^i u^2 \rangle = -4\epsilon. \)

4/5 law: \( \langle u_i^3 \rangle = -\frac{4}{5} \epsilon r. \)

\[ \left\langle u^2 \frac{u_l}{r} \right\rangle \propto \epsilon \]

Energy at scale \( r \)  Typical rate at scale \( r \)
The scaling of velocity structure functions:
\[ \langle u_l^p \rangle \propto r^{\zeta_p} \]
\[ \zeta_p = \frac{p}{3} \]

Gotoh et. al. 2002
WHERE TO FIND TWO DIMENSIONAL TURBULENCE?

Experiments – thin fluid layers:


CASCADES IN TURBULENCE

3d:

2d:
In absence of dissipation, energy and (any power of) vorticity are conserved,

\[
2E = \int \tilde{v}^2 \, d^2x = \int |v_k|^2 \, d^2k \\
2\Omega = \int (\nabla \times \tilde{v})^2 \, d^2x = \int \omega^2 \, d^2x = \int k^2 |v_k|^2 \, d^2k
\]
In absence of dissipation, energy and (any power of) vorticity are conserved,

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Dissipation occurs at small scales (large k):

\[ \partial_t \tilde{v} + \tilde{v} \cdot \nabla \tilde{v} = -\nabla p + \nu \nabla^2 \tilde{v} ; \quad \nabla \cdot \tilde{v} = 0 \]
CASCADeS IN 2D TURBuLENCE

\[ 2E = \int \nu^2 \, d^2x = \int \left| \nu_k \right|^2 \, d^2k \]

\[ 2\Omega = \int (\nabla \times \nu)^2 \, d^2x = \int \omega^2 \, d^2x = \int k^2 \left| \nu_k \right|^2 \, d^2k \]

As \( \nu \to 0 \) the dissipation scale \( k_d \to \infty \)

If \( \left| \nu_{k_d} \right|^2 \to \text{finite} \), then \( k_d^2 \left| \nu_{k_d} \right|^2 \to \infty \)

More enstrophy dissipated than what was created
INERTIAL RANGE

2d:

IR cutoff: box scale
UV cutoff: forcing scale
In the inertial range of scales:

\[ \nabla^i \langle u^i u^2 \rangle = 4\epsilon. \]

\[ \langle \nu \cdot f \rangle = \epsilon \]
2D TURBULENCE IN FINITE BOX – A CONDENSATE

\[ \partial_t \tilde{v} + \nabla \cdot \nabla \tilde{v} = -\nabla p + \nu \nabla^2 \tilde{v} + f - \alpha \tilde{v} ; \quad \nabla \cdot \tilde{v} = 0 \]

Shats, Xia, Punzmann, PRE (2005)

\( l_f \) - forcing correlation length
2D TURBULENCE IN FINITE BOX – A CONDENSATE

Shats, Xia, Punzmann, PRE (2005)

\[ \partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \nu \nabla^2 \vec{v} + f - \alpha \vec{v} ; \quad \nabla \cdot \vec{v} = 0 \]

\( \alpha \): bottom drag - large scale dissipation mechanism
2D TURBULENCE IN FINITE BOX – A CONDENSATE

\[ \partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \nu \nabla^2 \vec{v} + f - \alpha \vec{v} ; \quad \nabla \cdot \vec{v} = 0 \]

Understand the condensate and fluctuations around it

Shats, Xia, Punzmann, PRE (2005)
LARGE SCALE MEAN FLOW

(earth.nullschool.net)
SIMULATIONS

Vorticity:

Courtesy of Jason Laurie
• Forcing at small scales, $l_f \ll L$

• White in time forcing, $\langle \hat{f} \rangle = 0$

• Kinetic energy injection rate $\langle \hat{v} \cdot \hat{f} \rangle = \epsilon$

Vorticity:

Courtesy of Jason Laurie
NON DIMENSIONAL PARAMETERS

- $\delta = \alpha T_{\text{turnover}} = \alpha \epsilon^{-1/3} L^{2/3} \ll 1$

- $K = \frac{L}{l_f} \gg 1$

- $\Gamma = \frac{\nu l_f^{-2}}{\alpha}$

(Re = $\delta^{-1} \Gamma^{-1} K^{2/3} \gg 1$)
\[
\vec{v} = (U + u, v)
\]

\[
\langle \vec{v} \rangle = (U, 0)
\]

Average over time/space

$U(r)$

Mean Flow

$u, v$

Velocity Fluctuations

(EASIER) NOTATIONS

Mean Flow

\( U(y) \)

Velocity Fluctuations

\( u \)

\( v \)
TOTAL ENERGY BALANCE

\[ \int \alpha U^2 d^2x = \epsilon L^2 \]

Dissipation rate  \quad Injection rate

\[ U \sim \sqrt{\frac{\epsilon}{\alpha}} \quad u, v \ll U \]
TOTAL ENERGY BALANCE

\[ \int \alpha U^2 d^2x = \epsilon L^2 \]

Dissipation rate \hspace{1cm} Injection rate

\[ U \sim \sqrt{\frac{\epsilon}{\alpha}} \quad u, v \ll U \]

Looking for an expression for \( U, \langle uv \rangle, \langle u^2 \rangle, \langle v^2 \rangle \) ...
TOTAL ENERGY BALANCE

\[ \int \alpha U^2 d^2x = \epsilon L^2 \]

Dissipation rate \hspace{2cm} Injection rate

\[ U \sim \sqrt[2]{\frac{\epsilon}{\alpha}} \quad u, v \ll U \]

Looking for an expression for \( U, \langle uv \rangle, \langle u^2 \rangle, \langle v^2 \rangle \) ...
FROM THE MOMENTUM FLUX TO THE MEAN VELOCITY

• Momentum balance in x direction: \( \partial_y \langle uv \rangle + \alpha U = 0 \)

• Energy balance for fluctuations: \( \partial_y \langle v\left[\frac{(u^2+v^2)}{2} + p\right]\rangle = -U'\langle uv \rangle + \epsilon - D \)
FROM THE MOMENTUM FLUX TO THE MEAN VELOCITY

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Spatial energy flux

Energy transfer between mean and fluctuations
FROM THE MOMENTUM FLUX TO THE MEAN VELOCITY

- Momentum balance in x direction: \( \partial_y \langle uv \rangle + \alpha U = 0 \)

- Energy balance for fluctuations: \( \partial_y \langle v \left[ \frac{u^2 + v^2}{2} + p \right] \rangle = -U' \langle uv \rangle + \epsilon - D \)

\[ 0 = -U \partial_y \langle uv \rangle - \alpha U^2 \]
FROM THE MOMENTUM FLUX TO THE MEAN VELOCITY

- Momentum balance in x direction: \( \partial_y \langle uv \rangle + \alpha U = 0 \)

- Energy balance for fluctuations: \( \partial_y \langle v\left[\frac{u^2 + v^2}{2} + p\right]\rangle = -U'\langle uv \rangle + \epsilon \)

\( u, v \ll U \)

Quasi- linear approximation: \( \partial_y \langle vp \rangle = -U'\langle uv \rangle + \epsilon \)
FROM THE MOMENTUM FLUX TO THE MEAN VELOCITY

• Momentum balance in x direction: \( \partial_y \langle uv \rangle + \alpha U = 0 \)

• Quasi-linear approximation: \( \partial_y \langle vp \rangle = -U' \langle uv \rangle + \epsilon \)

Shear rate \( U' \) faster than non-linear interaction rate at \( k_f \)
• In absence of forcing and dissipation, the system has the PT symmetry

\[ x \to -x, \ t \to -t \]

Separately, are a symmetry of the Euler equation

\[ U \text{ and its derivatives are invariant w.r.t it} \]
• In absence of forcing and dissipation, the system has the PT symmetry

\[ x \rightarrow -x, \ t \rightarrow -t \]

• The momentum flux \( \langle uv \rangle \) is odd with respect to PT
• In absence of forcing and dissipation, the system has the PT symmetry

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• The momentum flux \[ \langle uv \rangle \] is odd with respect to PT

• \[ \langle vp \rangle \] is odd with respect to PT
• In absence of forcing and dissipation, the system has the PT symmetry

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• The momentum flux \( \langle uv \rangle \) is odd with respect to PT

The forcing breaks the symmetry, makes \( \langle uv \rangle \neq 0 \)
• In absence of forcing and dissipation, the system has the PT symmetry

\[ x \rightarrow -x, \ t \rightarrow -t \]

• The momentum flux \(\langle uv \rangle\) is odd with respect to PT

The forcing breaks the symmetry, makes \(\langle uv \rangle \neq 0 \quad \langle uv \rangle \propto \epsilon\)
The only remaining time scale is $U'$, so that

$$\langle uv \rangle \propto \frac{\epsilon}{U'} \quad ([\epsilon] = [cm^2/sec^3])$$
The only remaining time scale is $U'$, so that

$$\langle uv \rangle \propto \frac{\epsilon}{U'}$$

$([\epsilon] = [cm^2/sec^3])$

Recall

The energy balance:

$$\partial_y \langle vp \rangle = -U' \langle uv \rangle + \epsilon$$
The only remaining time scale is $U'$, so that

$$\langle uv \rangle \propto \frac{\epsilon}{U'} \quad ([\epsilon] = [cm^2/sec^3])$$

$$\partial_y \langle vp \rangle = -U' \langle uv \rangle + \epsilon$$

Due to PT symmetry, $\langle vp \rangle$ is also determined by the forcing
AN ADDITIONAL SYMMETRY

- The force acts at small scales, $K \gg 1$

The dynamics determining $\langle uv \rangle, \langle vp \rangle$ is local, controlled by the shear rate $U'$
• The force acts at small scales, $K \gg 1$

The dynamics determining $\langle uv \rangle, \langle vp \rangle$ is local, controlled by the shear rate $U'$

If the pumping is statistically homogeneous in $y$ then:

\[ x \to -x, \quad y \to -y \]

is a symmetry of the statistics of $\langle uv \rangle, \langle vp \rangle$ at each point
If the \textbf{pumping is statistically homogeneous in y} then:

\[ x \rightarrow -x, \quad y \rightarrow -y \]

is a \textbf{symmetry} of \( \langle uv \rangle, \langle vp \rangle \):

It’s a symmetry of the Euler equation

\[ U' \rightarrow U' \]
If the force is statistically homogeneous in $y$ then:

\[ x \rightarrow -x, \quad y \rightarrow -y \]

is a symmetry of $\langle uv \rangle, \langle vp \rangle$

However,

\[ \langle uv \rangle \rightarrow \langle uv \rangle \quad \text{while} \quad \langle vp \rangle \rightarrow -\langle vp \rangle \]
\[ \langle uv \rangle \rightarrow \langle uv \rangle \quad \text{while} \quad \langle vp \rangle \rightarrow -\langle vp \rangle \]

Therefore \( \langle vp \rangle = 0 \) and

\[ 0 = \partial_y \langle vp \rangle = -U' \langle uv \rangle + \epsilon \]
CLOSED EQUATIONS FOR THE MEAN FLOW

\[ U'(uv) = \epsilon \]

\[ \partial_y(uv) + \alpha U = 0 \]
THE VORTEX PROFILE

\[ U' \langle uv \rangle = \epsilon \]
\[ \partial_y \langle uv \rangle + \alpha U = 0 \]
\[ r \langle uv \rangle \partial_r \frac{U}{r} = \epsilon \]
\[ r^{-1} \partial_r r^2 \langle uv \rangle + \alpha r U = 0 \]

For the vortex mean flow (polar coordinates):

\[ U = \sqrt{\frac{3\epsilon}{\alpha}} \]
\[ \langle uv \rangle = -r \sqrt{\frac{\epsilon \alpha}{3}} \]
\begin{itemize}
  \item Laurie, Boffetta, Falkovich, Kolokolov, Lebedev, PRL 2014
  \item Kolokolov, Lebedev PRE 2016
    \begin{itemize}
      \item AF Phys. Fluids, 2017
    \end{itemize}
\end{itemize}

\[ U = \sqrt{\frac{3\epsilon}{\alpha}} \quad \langle uv \rangle = -r \sqrt{\frac{\epsilon \alpha}{3}} \]
COMPARISON TO SIMULATIONS

AF, Herbert, PRL 2018
Laurie et al. PRL 2014
Looking for an expression for $U$, $\langle uv \rangle$, $\langle u^2 \rangle$, $\langle v^2 \rangle$
TWO-POINT CORRELATION FUNCTIONS
AND ENERGY

An expression for $U$, $\langle uv \rangle$, $\langle u^2 \rangle$, $\langle v^2 \rangle$

A single equation determines $\langle u_1 v_2 \rangle$, $\langle u_1 u_2 \rangle$, $\langle v_1 v_2 \rangle$ due to incompressibility
An expression for $U, \langle uv \rangle, \langle u^2 \rangle, \langle v^2 \rangle$

A single equation determines $\langle u_1 v_2 \rangle, \langle u_1 u_2 \rangle, \langle v_1 v_2 \rangle$

Navier-Stokes+steady state: $\mathfrak{L}(U)\langle v_1 v_2 \rangle = \langle v_2 f_1 \rangle + \langle v_1 f_2 \rangle + D$

$v_1 \equiv v(r_1, \phi_1)$

Advection operator
A single equation determines \(<u_1 v_2>, <u_1 u_2>, <v_1 v_2>\)

Navier-Stokes+steady state: \(\mathfrak{L}(U)<v_1 v_2> = <v_2 f_1> + <v_1 f_2>\)

\(v_1 \equiv v(r_1, \phi_1)\)
A single equation determines $\langle u_1 v_2 \rangle, \langle u_1 u_2 \rangle, \langle v_1 v_2 \rangle$

$$\mathcal{L}(U) \langle v_1 v_2 \rangle = \langle v_2 f_1 \rangle + \langle v_1 f_2 \rangle$$

$$v_1 \equiv v(r_1, \phi_1)$$

For $|\vec{r}_1 - \vec{r}_2| > l_f \rightarrow \langle v_2 f_1 \rangle = 0$
TWO-POINT CORRELATION FUNCTIONS
AND ENERGY

An expression for $U$, $\langle uv \rangle$, $\langle u^2 \rangle$, $\langle v^2 \rangle$

The dependence on the amplitude of $U$ drops out:

$$[L_2 r_2 (2 \partial_{r_1} r_1 + L_1) - L_1 r_1 (2 \partial_{r_2} r_2 + L_2)] \partial_{\phi_1} \langle v_1 v_2 \rangle = 0$$

$$L_i = r_i^2 \Delta_i$$

Linear, homogeneous 5th order PDE
TWO-POINT CORRELATION FUNCTIONS AND ENERGY

An expression for $U$, $\langle uv \rangle$, $\langle u^2 \rangle$, $\langle v^2 \rangle$

The dependence on the amplitude of $U$ drops out:

$$[L_2 r_2 (2 \partial_{r_1} r_1 + L_1) - L_1 r_1 (2 \partial_{r_2} r_2 + L_2)] \partial \phi_1 \langle v_1 v_2 \rangle = 0$$

$L_i = r_i^2 \Delta_i$

The system is isotropic:

A set of equations for the angular harmonics $\langle \hat{v}_m (r_1) \hat{v}_{-m} (r_2) \rangle$
TWO-POINT CORRELATION FUNCTIONS AND ENERGY

An expression for $U$, $\langle uv \rangle$, $\langle u^2 \rangle$, $\langle v^2 \rangle$.

The dependence on the amplitude of $U$ drops out:

$$[L_2 r_2 (2 \partial r_1 + L_1) - L_1 r_1 (2 \partial r_2 + L_2)] \partial \phi_1 \langle v_1 v_2 \rangle = 0$$

$L_i = r_i^2 \Delta_i$

The system is isotropic:

A set of equation for the angular harmonics $\langle \hat{v}_m(r_1) \hat{v}_{-m}(r_2) \rangle$

$$\langle v^2(r) \rangle = 2 \sum_{m \geq 0} \langle |\hat{v}_m(r)|^2 \rangle, \quad \langle u^2(r) \rangle = 2 \sum_{m \geq 0} \langle |\hat{u}_m(r)|^2 \rangle$$
TWO-POINT CORRELATION FUNCTIONS AND ENERGY

An expression for $U$, $\langle uv \rangle$, $\langle u^2 \rangle$, $\langle v^2 \rangle$

The dependence on the amplitude of $U$ drops out:

$$[L_2 r_2 (2 \partial_{r_1} r_1 + L_1) - L_1 r_1 (2 \partial_{r_2} r_2 + L_2)] \partial_{\phi_1} \langle v_1 v_2 \rangle = 0$$

$L_i = r_i^2 \Delta_i$

The ansatz $\langle \hat{v}_m (r_1) \hat{v}_m (r_2) \rangle = r_1^\lambda f_m \left( \frac{r_2}{r_1} \right)$ reduces the equation to a hypergeometric equation.
The ansatz $\left< \hat{v}_m(r_1) \hat{v}_{-m}(r_2) \right> = r_1^\lambda f_m \left( \frac{r_2}{r_1} \right)$ reduces the equation to a hypergeometric equation.

Need boundary conditions to determine $\lambda$ and the amplitudes.
FAMILY OF SOLUTIONS

- The ansatz \( \langle \hat{v}_m (r_1) \hat{v}_{-m} (r_2) \rangle = r_1^\lambda f_m \left( \frac{r_2}{r_1} \right) \) reduces the equation to a hypergeometric equation.

- Need boundary conditions to determine \( \lambda \) and the amplitudes.
\[ \langle v^2(r) \rangle = 2 \sum_{m \geq 0} \langle |\hat{v}_m(r)|^2 \rangle, \quad \langle u^2(r) \rangle = 2 \sum_{m \geq 0} \langle |\hat{u}_m(r)|^2 \rangle \]

- Empirically, 90% of the energy is in the \( m=1 \) mode
FAMILY OF SOLUTIONS – USING SIMULATIONS

\[ \langle u^2 \rangle, \langle v^2 \rangle \]
TURBULENT ENERGY PROFILE: M=1 MODE

\[ \langle |\hat{v}_1|^2 \rangle = (\epsilon R_u)^{2/3} \left[ A_1 + A_2 \left( \frac{R_u}{L} \right)^{4/3} \left( \frac{l_f}{r} \right)^2 \right] \]

\[ \langle |\hat{u}_1|^2 \rangle = (\epsilon R_u)^{2/3} A_1 \quad \langle \hat{u}_1 \hat{v}_{-1} \rangle = i(\epsilon R_u)^{2/3} A_1 \]

\[ R_u = \alpha^{ \frac{1}{2} } l_f^{ \frac{2}{3} } \frac{1}{\epsilon^6} \]
COMPARISON TO SIMULATIONS: $M=1$ MODE
COMPARISON WITH SIMULATIONS – M=1 MODE
In 2d, turbulence leads to the formation of a coherent mean flow.

The presence of the mean flow allows for a consistent approximation, giving universal analytical results.

Symmetry dictates that the turbulent momentum flux and the turbulent energy are determined by different mechanisms.

References:
AF, Phys. Fluids 2017
AF C. Herbert, PRL 2018

Thank you for your attention! Questions?
\[ \langle |\hat{v}_2|^2 \rangle = (\epsilon l_f)^{2/3} \left[ B_1 - B_2 \left( \frac{R_u}{L} \right)^{4/3} \left( \frac{l_f}{r} \right)^2 \right] \]

\[ \langle \hat{u}_2 \hat{v}_{-2} \rangle = i(\epsilon l_f)^{2/3} \frac{1}{2} B_1 \]

\[ \langle |\hat{u}_2|^2 \rangle = (\epsilon l_f)^{2/3} \left[ \frac{19}{28} B_1 - \frac{\sqrt{3}}{2} B_2 \left( \frac{R_u}{L} \right)^{4/3} \left( \frac{l_f}{r} \right)^2 \right] \]

\[ R_u = \alpha^{-1/2} l_f^{2/3} \epsilon^{1/6} \]
$M=2$ MODE