Birth, Death, and Flight: A Theory of Malthusian Flocks
An “Immortal” Flock: Flamingos in Africa (~$10^5$ birds)

$L \sim 1$ km
A “Malthusian” flock: the Mitotic spindle
Outline

• I) Review of “Immortal” flocks (flocks with number conservation)
  (with Yu-hai Tu, IBM Watson)

• II) “Malthusian” flocks (number not conserved, due to “birth and death”)
  3d: Leiming Chen, Chiu Fan Lee, JT, in preparation

“The road to hell is paved with works in progress” (Philip Roth)
Immortal Flocks

I) Microscopic models (Vicsek)

   Important points: Rotation Invariance
   Locality

II) Mermin_Wagner Theorem: Are Birds smarter than nerds?

III) Continuum Theory: Analogy with Fluid Mechanics

IV) Predictions: How motion beats Mermin-Wagner
I) Microscopic Models

Vicsek algorithm:

Velocity of Central bird “i”

Velocity of Neighbors of central bird “j”

Errors

Update rule:

\[
\vec{V}_i
\]

\[
\vec{V}_j
\]

\[
\vec{v} (t + 1) = v_0
\]

\[
\sum_j \vec{v}_j(t) + \vec{f}_i(t + 1)
\]

\[
\sum_j \vec{v}_j(t) + \vec{f}_i(t + 1)
\]
Essential Features of Algorithm

• Only **Local** interactions: **short ranged in space and time**
• Ferromagnetic interactions (favor alignment)
• “Birds” *keep moving* ( $\vec{v} \neq \vec{0}$ ) and making errors

Symmetries and conservation laws
### Symmetries:

<table>
<thead>
<tr>
<th></th>
<th>Dynamics</th>
<th>Phase</th>
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<tbody>
<tr>
<td>Translation Invariance</td>
<td>YES</td>
<td>YES</td>
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<td></td>
<td>&lt; $\rho(\vec{r},t) \equiv \rho_0$</td>
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<td>$= CONSTANT$</td>
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<tr>
<td>Rotation Invariance</td>
<td>YES</td>
<td>NO</td>
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<td>$&lt; \vec{v}(\vec{r},t) \equiv \vec{v}_0 \neq \vec{0}$</td>
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<tr>
<td>Galilean Invariance</td>
<td>NO</td>
<td>NO</td>
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</table>
Conservation laws:

(For this part of the talk)
Bird number (density): Yes

Momentum: No (frictional substrate acts as a momentum sink)
(“Dry” versus “wet”)
Dynamics produces order:

\[ t=0 \]

\[ \langle \vec{v}(\vec{r}, t) \rangle = 0 \]

Run algorithm
Many steps (\( t \gg 1 \))

\[ t \gg 1 \]

\[ \langle \vec{v}(\vec{r}, t) \rangle = \vec{v}_0 \neq 0 \]
However.....

This should be Impossible!

Why?

Violates Mermin-Wagner theorem

Are Birds smarter than nerds?
Mermin-Wagner theorem:

Pointers vs. Flockers

- APS pointers ("XY model"):

\[ \langle \vartheta^2 \rangle = \Delta \ln(L) \]

as \( L \to \infty \)

No Long Ranged order

\[ \langle \tilde{S}(\vec{r},t) \rangle = 0 \]

Equilibrium result
Continuum Theory of Immortal Flocks

• Hard (impossible) to solve microscopic model with $\sim 10^{5}$ birds
• Harder to figure out what happens if you change model (universal vs system-specific)
• Historical analog: Fluid mechanics (Navier, Stokes, 1822):
  - No theory of atoms and molecules
  - No statistical physics
  - No computers, ipad, ipod, etc

• So, how’d they do it?
Continuum Approach

Replace \( \vec{r}_i(t) \) \( \rightarrow \) Continuous fields:

\[
\rho(\vec{r},t) : \text{Coarse grained number density}
\]

\[
\vec{v}(\vec{r},t) : \text{Coarse grained velocity}
\]

Valid for: Length scales \( L >> \) interatomic distance

Time scales \( t >> \) collision time
Why these fields?

Why only these fields?

They’re slow: lifetime \( T(L \rightarrow \infty) \rightarrow \infty \)

Why slow?

\( \rho(\vec{r}, t) \) Conserved

For this part of talk!

\( \vec{v}(\vec{r}, t) \) Broken symmetry (Goldstone mode)
Must move particles a distance $L$ to change $\rho \rightarrow \infty \Rightarrow \infty$: 

\[ T(L) \]
Broken symmetry $\Rightarrow$ slow
(Goldstone’s theorem)

Information must travel a distance $L$ to relax this distortion (because the symmetry is spontaneously broken) $\Rightarrow T(L\rightarrow\infty) \rightarrow \infty$
Why only these fields?

- Fast fields get **enslaved** to slow fields

I’ll show this explicitly for flocks with birth and death, in which $\rho$ becomes fast, because it’s not conserved

$$\rho = \mathcal{F} \left( \{v(r, t)\} \right)$$
For now, back to immortal flocks, $\rho$ conserved

Equations of motion for $\rho(\vec{r}, t), \vec{v}(\vec{r}, t)$

Make ‘em up!

Rules:  
- Lowest order in space, time derivatives
- Lowest order in fluctuations

\[
\delta \rho(\vec{r}, t) \equiv \rho(\vec{r}, t) - \langle \rho(\vec{r}, t) \rangle \\
\delta \vec{v}(\vec{r}, t) \equiv \vec{v}(\vec{r}, t) - \langle \vec{v}(\vec{r}, t) \rangle
\]

Respect Symmetries (for flocks, Rotation invariance)

Worked for fluids, should work for flocks
The Navier-Stokes Equations

\[
\rho \left( \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla P(\rho) + \eta_s \nabla^2 \mathbf{v} + \eta_B \nabla (\nabla \cdot \mathbf{v}) + \mathbf{f}
\]

“naïve acceleration”
“convective derivative”
“pressure”
“Shear viscosity”
“Bulk viscosity”
“noise” (models thermal Fluctuations)

Density EOM
\[
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0
\]

“Continuity equation”
(particle number conserved)
Our (Yu-hai Tu and JT) idea: same approach, different symmetry

- No **Galilean** invariance (birds move through a Special “rest frame” (e.g., air, water, surface of Serengeti. Etc....)))
Hydrodynamic equations for Immortal Flocks:

Density EOM:
\[
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0
\]

Velocity EOM:
\[
\begin{align*}
\partial_t \mathbf{v} + \lambda_1 (\mathbf{v} \cdot \nabla) \mathbf{v} + \lambda_2 \mathbf{v} (\nabla \cdot \mathbf{v}) + \lambda_3 (\nabla \mid \mathbf{v} \mid^2) &= \alpha \mathbf{v} - \beta \mid \mathbf{v} \mid^2 \mathbf{v} \\
-\nabla P(\rho) - \mathbf{v} (\mathbf{v} \cdot \nabla P_2(\rho)) + D_B \nabla (\mathbf{v} \cdot \nabla) \mathbf{v} + D_T \nabla^2 \mathbf{v} + D_2 (\mathbf{v} \cdot \nabla)^2 \mathbf{v} &= f
\end{align*}
\]

New terms (forbidden in NS equations due to Galilean invariance):

- Other "2 V dot grad" terms - Slowpoke "but not TOO Fast!"
- "2 V dot V" terms - "move fast but not"
- "convective Derivative""

Anisotropic pressure

Anisotropic viscosity

Number conservation ("immortal" flock)
\vec{f}(\vec{r}, t) : \text{Langevin White Noise}
 Acceleration in direction of motion:

\[ \hat{v} \cdot \partial_t \vec{v} \]

\[ \alpha(|v|) |v| - \beta(|v|)|v|^3 \]

\[ \Rightarrow <\vec{v}(\vec{r}, t) >= v_0 \hat{x} \]

Arbitrary direction (spontaneously broken symmetry)
Predictions of hydrodynamic theory

• Sound waves
• Giant number fluctuations
• Anomalous Hydrodynamics for $d<4$.
• Long-ranged order in $d=2$
Malthusian Flocks: Flocks with Birth and Death during flight

• Example: Mitotic spindle
• Homage to Malthus
• Hydrodynamic model
• Predictions:
  • Still LRO ($< \vec{v} > \neq \vec{0}$) even in $d=2$.
  • (And it outlives the birds!)
  • No sound waves
  • No Giant Number fluctuations
  • But persistent number fluctuations
Mitotic spindle: Mechanism of cell reproduction

DNA

Microtubules
"The power of population is so superior to the power of the earth to produce subsistence for man, that premature death must in some shape or other visit the human race. The vices of mankind are active and able ministers of depopulation. They are the precursors in the great army of destruction, and often finish the dreadful work themselves. But should they fail in this war of extermination, sickly seasons, epidemics, pestilence, and plague advance in terrific array, and sweep off their thousands and tens of thousands. Should success be still incomplete, gigantic inevitable famine stalks in the rear, and with one mighty blow levels the population with the food of the world".

\[ g(\rho) \]

Population growth rate (birth-death)

Steady state density

Population density
Hydrodynamic equations for Malthusian flocks

Velocity EOM:

\[ \partial_t \vec{v} + \lambda_1 (\vec{v} \cdot \vec{\nabla}) \vec{v} + \lambda_2 \vec{v} (\vec{\nabla} \cdot \vec{v}) + \lambda_3 (\vec{\nabla} |\vec{v}|^2) = \alpha \vec{v} - \beta |\vec{v}|^2 \vec{v} \]
\[ -\vec{\nabla} P(\rho) - \vec{v} (\vec{v} \cdot \vec{\nabla} P_2(\rho)) + D_B \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) + D_T \nabla^2 \vec{v} + D_2 (\vec{v} \cdot \vec{\nabla})^2 \vec{v} + \vec{f} \]

(Same as for immortal flocks)

Density EOM:

\[ \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = g(\rho) \]
"Malthusian" form of $g(\rho)$

Tangent at $\rho_0$, Slope $= -\frac{1}{\tau}$

$\tau = \text{Population relaxation time} \sim \text{critter lifetime} \sim 1 \text{ generation}$
Does population density relax to $\rho_0$ in time $\tau$?

No! To higher (lower) population set by immigration (emigration)

Negligible
(slow, hydrodynamic mode)

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = g(\rho) \approx -\frac{(\rho - \rho_0)}{\tau}$$

⇒ $\rho = \rho_0 (1 - \tau \nabla \cdot \vec{v})$ : $\rho$ Enslaved to $\vec{v}$
This is what causes Giant Number Fluctuations in “Immortal” Flocks; here, it’s suppressed by increased death rate (Malthus was right!)
Substitute in $v$ EOM, get closed EOM for $v$ alone:

$$
\frac{\partial}{\partial t} \bar{v} + \lambda_1 (\bar{v} \cdot \bar{\nabla}) \bar{v} + \lambda_2 \bar{v} (\bar{\nabla} \cdot \bar{v}) + \lambda_3 (\bar{\nabla} |\bar{v}|^2) = \alpha \bar{v} - \beta |\bar{v}|^2 \bar{v}
$$

$$
+ D_B' \bar{\nabla} (\bar{\nabla} \cdot \bar{v}) + D_T \nabla^2 \bar{v} + D_2 (\bar{v} \cdot \bar{\nabla})^2 \bar{v} + \tilde{f}
$$

$$
\rho \propto \bar{\nabla} \cdot \bar{v}
$$
Expand in fluctuations:

\[ \mathbf{v}(\mathbf{r}, t) = (v_0 + u_{||})\hat{x} + u_{\perp}(\mathbf{r}, t) \]

\[ \mathbf{v}(\mathbf{r}, t) \]

Fast => Eliminate
Gives theory for:

Linear

\[ \sim |r_\perp|^{\chi-z} \]

Linear

Nonlinear

Power counting:

Dynamic exponent \( z \):

\[ \langle f_i(r,t)f_j(r',t') \rangle = 2D\delta_{ij}\delta^d(r-r')\delta(t-t') \]
\[ \sim \delta^{d-1}(r_\perp-r'_\perp)\delta(x-x')\delta(t-t') \]
\[ \sim x^{-1}|r_\perp|^{-(d-1)t-1} \]

\[ f \sim \sqrt{x^{-1}|r_\perp|^{-(d-1)t-1}} \sim |r_\perp|^{(1-d-z-\zeta)/2} \]

\[ u_\perp \sim |r_\perp|^{\chi} \]

\[ u_\perp \sim |r_\perp|^{(1-d-z-\zeta)/2} \]
Linear theory scaling laws:

Equate powers of $|\mathbf{r}_\perp|$ of linear terms:

\[
\partial_t \mathbf{u}_\perp + \lambda (\mathbf{u}_\perp \cdot \nabla_\perp) \mathbf{u}_\perp = \mu_1 \nabla^2_\perp \mathbf{u}_\perp + \mu_2 \nabla_\perp (\nabla_\perp \cdot \mathbf{u}_\perp) + \mu_x \partial_x^2 \mathbf{u}_\perp + \mathbf{f}_\perp
\]

\[
\sim |\mathbf{r}_\perp|^{\chi - 2}
\]

\[
\sim |\mathbf{r}_\perp|^{(1-d-z-\zeta)/2}
\]

\[
\sim |\mathbf{r}_\perp|^{\chi - 2}\zeta
\]

\[
\sim |\mathbf{r}_\perp|^{\chi - 2\zeta}
\]

Note: $\chi = 0$ in $d=2$

\[
u_\perp \sim \ln(|\mathbf{r}_\perp|) \rightarrow \infty \Rightarrow \text{No LRO}
\]

(Mermin-Wagner Thm)
Can Malthusian Flock in \( d=2 \) have LRO? Yes! Nonlinearity to the rescue!

\[
\partial_t \mathbf{u}_\perp + \lambda (\mathbf{u}_\perp \cdot \nabla_\perp) \mathbf{u}_\perp = \mu_1 \nabla_\perp^2 \mathbf{u}_\perp + \mu_2 \nabla_\perp (\nabla_\perp \cdot \mathbf{u}_\perp) + \mu_x \partial_\perp^2 \mathbf{u}_\perp + f_\perp
\]

\[
\sim |\mathbf{r}_\perp|^{x-z}
\]

\[
\sim |\mathbf{r}_\perp|^{2x-1}
\]

\[
\mathbf{u}_\perp \sim |\mathbf{r}_\perp|^x
\]

\[
\text{ratio} \sim |\mathbf{r}_\perp|^{x+z-1} = |\mathbf{r}_\perp|^{(4-d)/2} \to \infty
\]

\[
z = 2
\]

\[
\zeta = 1
\]

\[
\chi = (2 - d)/2
\]

\[\Rightarrow\] Nonlinearity changes scaling for \( d<4! \)
So what *is* new true scaling for $d<4$? $d=2$ is simple:

JT, PRL 108, 088102 (2012)

$$(u_{\perp} \cdot \nabla_{\perp}) u_{\perp} \rightarrow u_y \partial_y u_y = \frac{1}{2} \partial_y (u_y^2)$$

Total $y$ derivative!

$$\partial_t u_y + \frac{\lambda}{2} \partial_y (u_y^2) = (\mu_1 + \mu_2) \partial_y u_y + \mu_x \partial_x u_y + f_y$$
Nonlinear power counting:

\[ \sim |r_\perp|^{\chi-z} \]

Neglect (dominated by NL term)

\[ \sim |r_\perp|^{(1-d-z-\zeta)/2} \]

3 linear equations, 3 unknowns: solution:

\[ \chi - z = 2\chi - 1 \]
\[ \chi - z = \chi - 2\zeta \Rightarrow z = 2\zeta \]
\[ (1 - d - z - \zeta)/2 = \chi - z \]

\[ z(d = 2) = \frac{2(d+1)}{5} = \frac{6}{5} \]
\[ \zeta(d = 2) = \frac{d+1}{5} = \frac{3}{5} \]
\[ \chi(d = 2) = \frac{3 - 2d}{5} = -\frac{1}{5} \]
\[ \chi = -1/5 < 0 \Rightarrow \text{long-ranged order!} \]

\[ < \tilde{v} > \neq 0 \]

And it outlives the birds!

Persistence time \( T \) of \( < \tilde{v}(\tilde{r}, t) > \) \( \gg \tau \) Lifetime Of one bird

\[ T \propto N \quad \text{(number of birds)} \quad \gg \tau \quad \text{Independent of N} \]
No sound modes; instead, convention
+ (hyper) diffusion

Diffusion along mean flock motion

$\sqrt{t} \hat{x}$

Initial disturbance (e.g., high density)

$(\text{hyper})$ diffusion
Perpendicular to Mean flock motion

$t^{5/6}$

$\langle \bar{v} \rangle \equiv v_0 \hat{x}$

$v_2 \neq v_0$
Persistent number fluctuations

\[ < \rho(\vec{r}, t) \rho(\vec{r} + v_2 t \hat{x}, t = 0) > \propto t^{-2} \]

Initial disturbance (e.g., high density)

\[ < \rho(\vec{r}, t) \rho(\vec{r}, t = 0) > \propto t^{-4} \]

\[ < \vec{v} > \equiv v_0 \hat{x} \]
What about $d=3$?

With Chiu Fan Lee, Imperial college, London

Leiming Chen, China University of Mining and Technology

$$(u_\perp \cdot \nabla_\perp) u_\perp$$ is not a total derivative

$\Rightarrow$ No simple power counting argument

$\Rightarrow$ Must do full blown dynamical RG expansion

$\epsilon = 4 - d$
We thought it would be easy: \[ \text{WRONG!} \]

Nonlinear coupling:

\[
\begin{align*}
1 \frac{dg_1}{g_1 \, dl} &= \frac{5}{2} G_{\mu_1} \equiv \frac{D \lambda^2}{\sqrt{\mu_1 \mu_2} (2\pi)^{d-1}} A^{d-4}, \quad g_1 \equiv \frac{\mu_2}{\mu_1} \\
\frac{1}{g_2} \frac{dg_2}{dl} &= g_1 (G_{\mu_2} - G_{\mu_1}) \equiv g_1 G_{g_2}(g_2),
\end{align*}
\]

\[
G_{g_1}(g_2) = \frac{(-10d^2 + 30d - 15)}{32(d^2 - 1)} + \frac{(d^2 - d + 8)}{2(d^2 - 1)g_2^2} - \frac{(2d^2 + 3d + 11)\sqrt{2}}{(d^2 - 1)g_2^2(g_2 + 2)^{3/2}} + \frac{(d + 3)}{2(d - 1)g_2^2\sqrt{g_2 + 1}} + \frac{(15 - 5d)}{2(d^2 - 1)g_2} \\
&- \frac{\sqrt{2}(4d^2 - 9d + 97)}{4(d^2 - 1)g_2(g_2 + 2)^{3/2}} + \frac{(5d - 45)}{2\sqrt{2}(d^2 - 1)(g_2 + 2)^{3/2}} + \frac{5}{4(d - 1)g_2\sqrt{g_2 + 1}}
\]

\[
G_{g_2}(g_2) = \left(\frac{2}{d^2 - 1}\right) \left(\frac{(1 - 3d)\sqrt{2}}{g_2^3(g_2 + 2)^{3/2}} + \frac{(d - 1)}{g_2^2}\frac{d + 1}{2g_2\sqrt{g_2 + 1}} + \frac{(1 - 19d)}{2\sqrt{2}g_2^2(g_2 + 2)^{3/2}} + \frac{(d^2 - 4d + 3)}{2\sqrt{2}g_2^2\sqrt{g_2 + 2}} - \frac{(d^2 - 7d + 4)}{4g_2^2} \right. \\
&+ \frac{3(d + 1)}{4g_2^2\sqrt{g_2 + 1}} - \frac{3\sqrt{2}}{2(g_2 + 2)^{3/2}} - \frac{(9 + 7d)}{2\sqrt{2}g_2^2(g_2 + 2)^{3/2}} + \frac{(2d^2 - 25d + 59)}{32g_2} + \frac{d + 1}{64g_2(g_2 + 1)^{3/2}} + \frac{(d + 1)}{4g_2\sqrt{g_2 + 1}} \\
&\left. + \frac{(d - 3)}{2\sqrt{2}g_2\sqrt{g_2 + 2}} - \frac{(2d^2 - 6d + 3)}{32}\right)
\]
RG flows:

Non-linear coupling:

g_1
Results:

\[ z = 2 - \frac{6\epsilon}{11} + \mathcal{O}(\epsilon^2) \]
\[ \chi = -1 + \frac{6\epsilon}{11} + \mathcal{O}(\epsilon^2) \]
\[ \zeta = 1 - \frac{3\epsilon}{11} + \mathcal{O}(\epsilon^2) . \]

\[ z = \frac{28}{19} \approx 1.47 , \]
\[ \zeta = \frac{14}{19} \approx 0.74 , \]
\[ \chi = -\frac{9}{19} \approx -0.47 . \]
Summary:

• Flocks illustrate fundamental Hydrodynamic principles (symmetries, conservation laws)

• Birth and death turn off number conservation, which changes long length, time scale behavior of flocks radically (no sound, no GNF)

• But can still understand using hydrodynamic approach
Thanks for your attention!

And don’t have any more!

!@#$%^&*

Kids!!!!!!
Giant number fluctuations

Simulation by Markus Ulm

< \vec{v} >
Anomalous Hydrodynamics

Hydrodynamics With noise ≠ Hydrodynamics Without noise

Occurs for any \( d < 4 \) (i.e., \( d=2 \) and \( d=3 \))

In \( d=2 \): Only way to stabilize LRO!
Example: Sound damping:

Attenuation length $L_a$

Without Noise: $L_a \propto \lambda^2$

With Noise: $L_a \propto \lambda^{z(d)}$

$z(d)$ Universal

Same for all flocks in given $d$ But $d$-dependent, and $z(d<4)<2$
Why does this happen?

Fluctuations (waves) interact due to Convective nonlinearity and other nonlinearities from density:

\[ \partial_t \vec{v} + \lambda_1 (\vec{v} \cdot \nabla) \vec{v} + \lambda_2 \vec{v} (\nabla \cdot \vec{v}) + \lambda_3 (\nabla |\vec{v}|^2) = \alpha \vec{v} - \beta |\vec{v}|^2 \vec{v} \]

\[ -\nabla P(\rho) - \vec{v} (\vec{v} \cdot \nabla P_2(\rho)) + D_B \nabla (\nabla \cdot \vec{v}) + D_T \nabla^2 \vec{v} + D_2 (\vec{v} \cdot \nabla)^2 \vec{v} + \vec{f} \]
Calculating $z(d)$:

Nightmare!

In principle, straightforward dynamical RG calculation

But: Calculation only valid near $d=4$

Worse: Even that calculation is incredibly hard!
Expanded equations of motion:

\[ \partial_t \vec{v}_\perp + \gamma \partial_{\parallel} \vec{v}_\perp + \lambda^0_1 (\vec{v}_\perp \cdot \vec{\nabla}_\perp) \vec{v}_\perp = -g_1 \delta \rho \partial_{\parallel} \vec{v}_\perp - g_2 \vec{v}_\perp \partial_{\parallel} \delta \rho - \frac{c_0^2}{\rho_0} \vec{\nabla}_\perp \delta \rho - g_3 \vec{\nabla}_\perp (\delta \rho^2) \]

\[ + D_B \vec{\nabla}_\perp (\vec{\nabla}_\perp \cdot \vec{v}_\perp) + D_T \vec{\nabla}^2 \vec{v}_\perp + D_{\parallel} \partial^2_{\parallel} \vec{v}_\perp + \nu_t \partial_t \vec{\nabla}_\perp \delta \rho + \nu_{\parallel} \partial_{\parallel} \vec{\nabla}_\perp \delta \rho + \vec{f}_\perp \]

(2.18)

\[ \partial_t \delta \rho + \rho_o \vec{\nabla}_\perp \cdot \vec{v}_\perp + w_1 \vec{\nabla}_\perp \cdot (\vec{v}_\perp \delta \rho) + \nu_2 \partial_{\parallel} \delta \rho = D_{\rho_{\parallel}} \partial^2_{\parallel} \delta \rho + D_{\rho_{\perp}} \vec{\nabla}^2 \delta \rho + D_{\rho v} \partial_{\parallel} (\vec{\nabla}_\perp \cdot \vec{v}_\perp) \]

\[ + \phi \partial_t \partial_{\parallel} \delta \rho + w_2 \partial_{\parallel} (\delta \rho^2) + w_3 \partial_{\parallel} (|\vec{v}_\perp|^2), \]
Feynmann graphs:

Number of graphs=$(7^2) \times 3 + (7^3) \times 3 = 1176!
Long-ranged order in $d=2$

- Stabilized by this breakdown of hydrodynamics (damps out noise induced fluctuations (negative feedback)
- $z(2)<2 \Rightarrow$ faster damping of fluctuations
- $\Rightarrow$LRO
- We know this happens because we know $z(2)<2$, even though we don’t know it’s actual value
But how to determine scaling laws?

Consider Incompressible flock: six of seven NL’s involve $\rho$

$\Rightarrow$ All gone if we fix $\rho$

Can actually get exact scaling exponents in all d in this case (with very little work!)
Definitions of other exponents:

Anisotropy exponent:

\[ L_a \propto \lambda^{z(d)/\zeta(d)} = \lambda^2 \]

Roughness exponent:

\[ z = \frac{2(d+1)}{5}, \quad \zeta = \frac{d+1}{5}, \quad \chi = \frac{3-2d}{5} \]
\[ G_D(g_2) \equiv \frac{(d - 2)}{2(d - 1)} \frac{1}{g_2^2} \left[ 1 + \frac{1}{\sqrt{g_2 + 1}} - \frac{2\sqrt{2}}{\sqrt{g_2 + 2}} \right] \]
\[ = \frac{1}{g_2^2} \left[ \frac{1}{3} + \frac{1}{3\sqrt{g_2 + 1}} - \frac{2\sqrt{2}}{3\sqrt{g_2 + 2}} \right] , \quad (d = 4) \]

\[ G_{\mu_1}(g_2) \equiv \frac{2}{d^2 - 1} \left( \frac{2d^2 - 6d + 3}{32} + \frac{(d + 3)\sqrt{2}}{g_2^2(g_2 + 2)^{3/2}} - \frac{1}{g_2^2} - \frac{d + 1}{2g_2^2\sqrt{g_2 + 1}} + \frac{d - 3}{2g_2} + \frac{d + 15}{2\sqrt{2}g_2(g_2 + 2)^{3/2}} \right) \]
\[ + \frac{3}{\sqrt{2}(g_2 + 2)^{3/2}} - \frac{d + 1}{4g_2\sqrt{g_2 + 1}} + \frac{3 - d}{2\sqrt{2}g_2\sqrt{g_2 + 2}} \right) \]

\[ G_{\mu_2}(g_2) \equiv \frac{2}{(d^2 - 1)g_2} \left( - \frac{(3d - 1)\sqrt{2}}{g_2^2(g_2 + 2)^{3/2}} + \frac{(d - 1)}{g_2^2} + \frac{(d + 1)}{2g_2^2\sqrt{g_2 + 1}} + \frac{(d^2 - 4d + 3)\sqrt{2}}{4g_2\sqrt{g_2 + 2}} - \frac{(d^2 - 7d + 8)}{4g_2} \right) \]
\[ + \frac{d + 1}{64(g_2 + 1)^{3/2}} + \frac{(13 - 15d)}{2\sqrt{2}g_2(g_2 + 2)^{3/2}} - \frac{3(d - 1)}{\sqrt{2}(g_2 + 2)^{3/2}} + \frac{(d + 1)}{4g_2\sqrt{g_2 + 1}} + \frac{2d^2 - 9d + 11}{32} \right) \]