Fluid dynamics, quark-gluon plasma, and quasinormal spectra of black holes

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HYDRODYNAMICS AT ALL LENGTH-SCALES
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Motivation: ongoing experimental programs and theoretical advances of the last two decades

Experiments:

Experiments on heavy ion collisions at RHIC (2000-current) and LHC (2010-current) (relativistic, “many-body”, strongly interacting, non-equilibrium “hot” system)

Experimental realization (1995-1999) of new classes of quantum “many-body” systems (e.g. ultra-cold atomic Bose and Fermi gases), current extensive study of their collective behavior (non-relativistic, “many-body”, strongly interacting, non-equilibrium “cold” system)

Experiments with neutron and X-ray scattering of liquids and molecular dynamics simulations to study collective excitations in strongly correlated quantum systems

Theory:

Gauge-string duality: A “new” (1997) non-perturbative tool to study strongly interacting quantum systems (zero or finite temperature/density, relativistic and non-relativistic, equilibrium and non-equilibrium – but for limited class of theories/parameters)


Heavy ion collisions: RHIC/LHC
Heavy ion collisions: evolution of the quark-gluon plasma

\[ t_\ast \sim 10^{-24} \text{ sec} \]

<table>
<thead>
<tr>
<th>Event</th>
<th>Time ( t_\ast )</th>
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<tbody>
<tr>
<td>Initial state</td>
<td>( 0 )</td>
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<tr>
<td>Pre-equilibrium</td>
<td>( 1.2 t_\ast )</td>
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<td>QGP and</td>
<td>( (2 - 3) t_\ast )</td>
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<tr>
<td>hydrodynamic</td>
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<td>expansion</td>
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<td>Hadronization</td>
<td>( (4 - 6) t_\ast )</td>
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<tr>
<td>Hadronic gas</td>
<td>( 10 t_\ast )</td>
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Fluid dynamics is an effective theory valid in the long-wavelength, long-time limit

Fundamental degrees of freedom = densities of conserved charges

Equations of motion = conservation laws + constitutive relations\(^*\)

\[
\begin{align*}
\partial_a J^a &= 0 \\
J^i &= -D \nabla^i J^0 + \cdots
\end{align*}
\]

\[
\begin{align*}
\partial_t J^0 &= D \nabla^2 J^0 + \cdots \\
\omega &= -iDq^2 + \cdots
\end{align*}
\]

Example II

\[
\begin{align*}
\partial_a T^{ab} &= 0 \\
T^{ab} &= \varepsilon u^a u^b + P(\varepsilon) (g^{ab} + u^a u^b) + \Pi^{ab} + \cdots
\end{align*}
\]

* Modulo assumptions e.g. analyticity

** E.o.m. universal but transport coefficients depend on underlying microscopic theory
Consider relativistic neutral “conformal” fluid in a d-dimensional (curved) space-time

\[ T^{ab} = \varepsilon u^a u^b + P(\varepsilon) \left( g^{ab} + u^a u^b \right) + \Pi^{ab} + \cdots \]

Including only terms with first and second derivatives of fluid velocity:

\[ \Pi^{ab} = -\eta \sigma^{ab} \]

\[ + \eta \tau_\Pi \left[ \langle D \sigma^{ab} \rangle + \frac{1}{d-1} \sigma^{ab} (\nabla \cdot u) \right] \]

\[ + \kappa \left[ R^{ab} - (d-2) u_c R^{c \langle ab \rangle d} u_d \right] \]

\[ + \lambda_1 \sigma^{ac} \sigma^{b} c + \lambda_2 \sigma^{ac} \Omega^{b} c + \lambda_3 \Omega^{ac} \Omega^{b} c \]

Transport coefficients (in conformal case): \( \eta, \tau_\Pi, \kappa, \lambda_1, \lambda_2, \lambda_3 \)

Non-conformal case: 2 first order coefficients, 15 (10) second order coefficients (see S.Bhattacharyya, 1201.4654 [hep-th])
Notations used in the derivative expansion

\[ D \equiv u^a \nabla_a \]

\[ \Delta^{ab} \equiv g^{ab} + u^a u^b \]

\[ A\langle ab \rangle \equiv \frac{1}{2} \Delta^{ac} \Delta^{bd} (A_{cd} + A_{dc}) - \frac{1}{d-1} \Delta^{ab} \Delta^{cd} A_{cd} \equiv \langle A^{ab} \rangle \]

\[ \sigma^{ab} = 2\langle \nabla^a u^b \rangle \]

\[ \Omega^{ab} = \frac{1}{2} \Delta^{ac} \Delta^{bd} (\nabla_c u_d - \nabla_d u_c) \]

*Hydro definitions differ in the literature – see footnote 91 on page 128 of M.Haehl, R.Loganayagam, M.Rangamani, 1502.00636 [hep-th]
See Appendix B in S.Grozdanov, AOS, 1611.07053 [hep-th]
The “unreasonable effectiveness of hydrodynamics “…

A paraphrase (one of many) of Wigner’s “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” (1960)

Approach to equilibrium – expect:

\[ \langle T^{\mu\nu} \rangle \rightarrow \text{diag}(\varepsilon, P, P, P) \]

Dashed black lines – hydrodynamic approximation to different stress-tensor components

Hydrodynamics seems to work remarkably well even when the gradients are not small...
Beyond second order hydrodynamics

Tensors structures appearing in the derivative expansion have been analyzed using computer algebra in 1507.02461 [hep-th] by Grozdanov & Kaplis.

At third order, there are 20 relevant structures in the conformal case and 68 in the non-conformal one.

This still needs an entropy current analysis similar to the one in S.Bhattacharyya, 1201.4654 [hep-th]

Example: dispersion relations in conformal case

\[
\omega = -i \frac{\eta}{\varepsilon + P} k^2 - i \left[ \frac{\eta^2 \tau_\Pi}{(\varepsilon + P)^2} - \frac{\theta_1}{2(\varepsilon + P)} \right] k^4 + \cdots
\]

\[
\omega = \pm c_s k - i \Gamma k^2 \mp \frac{\Gamma}{2c_s} \left( \Gamma - 2c_s^2 \tau_\Pi \right) k^3 - i \left[ \frac{8\eta^2 \tau_\Pi}{9(\varepsilon + P)^2} - \frac{\theta_1 + \theta_2}{3(\varepsilon + P)} \right] k^4 + \cdots
\]

Here \( c_s = 1/\sqrt{3} \), \( \Gamma = \eta/(\varepsilon + P) \).
Hydrodynamic (gapless) modes

Small fluctuations of an equilibrium state (here: homogeneous, isotropic, neutral, relativistic) are the hydrodynamic modes with dispersion relations (in Fourier space $e^{-i\omega t + i q z}$):

Shear mode: \[ \omega = \omega(q) = -i \frac{\eta}{\epsilon + P} q^2 + \cdots \]

Sound mode: \[ \omega = \omega_{\pm}(q) = \pm v_s q - i \frac{\zeta + \frac{4}{3} \eta}{\epsilon + P} q^2 + \cdots \]

$\omega = \frac{\omega}{2\pi T}, \ q = \frac{q}{2\pi T}; \ \eta, \zeta$ - shear & bulk viscosities; $v_s$ - speed of sound; $\epsilon, P$ - energy density & pressure.

1) Do the series above converge? If so, what determines their radii of convergence? Does the effective theory “know” its limits? Why hydro is so effective at strong coupling?

$$\omega = \frac{p^2}{2m} - \frac{p^4}{8m^3} + \frac{p^6}{16m^5} + \cdots = \sqrt{p^2 + m^2} - m, \ p = \pm im$$

2) How do transport coefficients change when the coupling in an underlying microscopic theory changes? Can we interpolate between weak and strong coupling?

Thanks to holographic duality, these questions can be investigated for some QFTs.
Relativistic hydrodynamics to all orders

\[ \langle T^{\mu\nu} \rangle \equiv T^{\mu\nu}(x, t) = T_{eq}^{\mu\nu} + \delta T^{\mu\nu}(x, t), \quad T_{eq}^{\mu\nu} = \text{diag} (\epsilon, P, P, P) \]

Fundamental d.o.f. – densities of conserved charges: \( T^{00}, T^{0i} \)

All other components should be expressed through them (and their derivatives) via constitutive relations. To linear order in fluctuations, in Fourier space:

\[ \delta T^{ij} = -i A (q^i \delta T^{0j} + q^j \delta T^{0i}) + \delta T^{00} (B q^i q^j + C \delta^{ij}) + i q_k \delta T^{0k} (D q^i q^j + E \delta^{ij}) \]

Here \( A, B, C, D, E \) are functions of \( \omega, q^2 \) Expanding them around (0,0) to order k, we get the usual derivative expansion in k-th order hydrodynamics, e.g. for k=1:

\[ \delta T^{ij} = \delta^{ij} v_s^2 \delta T^{00} - \frac{i}{\epsilon + P} \left[ \eta \left( q^i \delta T^{0j} + q^j \delta T^{0i} - \frac{2}{3} \delta^{ij} q_k \delta T^{0k} \right) + \zeta \delta^{ij} q_k \delta T^{0k} \right] \]

Combining this with conservation equation \( \partial_\mu T^{\mu\nu} = 0 \), we get a 4x4 matrix (M) equation for fluctuations \( \delta T^{00}, \delta T^{0i} \)
Hydrodynamic spectral curve

Non-trivial solution for fluctuations $\delta T^{00}, \delta T^{0i}$: $\det M \equiv P(q^2, \omega) = 0$

For relativistic homogeneous & isotropic neutral fluid in $d+1$ dimensions

$$P(q^2, \omega) = (\omega + iq^2 \gamma_\eta(\omega, q^2))^{d-1} \left( \omega^2 + i\omega q^2 \gamma_s(\omega, q^2) - q^2 H(\omega, q^2) \right) = 0$$

Here the functions $\gamma_\eta, \gamma_s, H$ are simply related to $A,B,C,D,E$ in the constitutive relations

$$\delta T^{ij} = -iA \left( q^i \delta T^{0j} + q^j \delta T^{0i} \right) + \delta T^{00} \left( B q^i q^j + C \delta^{ij} \right) + iq_k \delta T^{0k} \left( D q^i q^j + E \delta^{ij} \right)$$

So we have 2 hydrodynamic spectral curves:

$$F_{\text{shear}} = \omega + iq^2 \gamma_\eta(\omega, q^2) = 0$$

$$F_{\text{sound}} = \omega^2 + i\omega q^2 \gamma_s(\omega, q^2) - q^2 H(\omega, q^2) = 0$$

We treat them as complex curves $F(x, y) = 0$ in the space of $(x \equiv q^2, y \equiv \omega) \in \mathbb{C}^2$

Note: at any finite order of the gradient expansion, $F(x,y)$ is a polynomial
Complex curves $F(x, y) = 0$ are interesting objects. Here, we are interested in solutions $y = y(x)$ and their properties (we assume the curve is analytic or algebraic).

Regular points: $F(x_*, y_*) = 0, F_y(x_*, y_*) \neq 0 \quad y = \sum_{n>n_0}^{\infty} a_n (x-x_*)^n$

Critical points: $F(x_*, y_*) = 0, F_y(x_*, y_*) = 0, \ldots F_y^{(p)}(x_*, y_*) \neq 0$

Puiseux series: $y = \sum_{n>n_0}^{\infty} a_n (x-x_*)^{\frac{n}{m_j}}, m_j = 1, \ldots p$

$$f(x, y) = y^5 - 4y^4 + 4y^3 + 2x^2y^2 - xy^2 + 2x^2y + 2xy + x^4 + x^3 = 0$$
Applying these theorems to hydrodynamic spectral curves

\[ F_{\text{shear}} = \omega + iq^2 \gamma_\eta(\omega, q^2) = 0 \]
\[ F_{\text{sound}} = \omega^2 + i\omega q^2 \gamma_s(\omega, q^2) - q^2 H(\omega, q^2) = 0 \]

we conclude that solutions are given by Taylor or Puiseux series converging in the vicinity of \( q=0 \)

Shear mode:
\[ \omega_\pm = -i \sum_{n=1}^{\infty} c_n (q^2)^n \]

Sound mode:
\[ \omega_\pm = -i \sum_{n=1}^{\infty} a_n e^{\pm \frac{i\pi n}{2}} (q^2)^{\frac{n}{2}} \]

Obstruction to the convergence of these series is the next critical point of the spectral curve.

Spectral curves can be easily found in theories with dual gravity description.
Example: Kepler’s equation at complex eccentricity

Newton proved in “Principia” that “all smooth ovals are algebraically non-integrable” – e.g. in Kepler’s Third Law one finds non-analyticity \( T \sim a^{3/2} \)

Kepler’s law of motion in parametric form (\( T \) – period, \( a \) – major semi-axis, \( e \) – eccentricity)

\[
\begin{align*}
r &= a (1 - e \cos \psi) \\
t &= \frac{T}{2\pi} (\psi - e \sin \psi)
\end{align*}
\]

Kepler’s equation:

\[
P = \tau - \psi + e \sin \psi = 0
\]

\[
\tau \equiv 2\pi t/T
\]

Series solution (Lagrange, 1771):

\[
\psi(\tau, e) = \tau + \sum_{n=1}^{\infty} a_n(\tau) \frac{e^n}{n!}
\]

The series converges for \(|e| \leq e_L \approx 0.662743\ldots \ \forall \tau \) (Laplace, 1823)
Kepler’s equation at complex eccentricity (continued)

Critical points of the Kepler curve: \[ P = \tau - \psi + e \sin \psi = 0 \]

\[ \frac{\partial P}{\partial \psi} = e \cos \psi - 1 = 0 \]

The critical point closest to the origin is: \[ e_c(\tau = \frac{\pi}{2}) \approx \pm 0.662743i \]
Another effective theory...

\[ S_{grav} = \int d^4x \sqrt{|g|} \left[ \Lambda + \frac{1}{16\pi G} R + a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \ldots \right] \]

Low-energy gapless d.o.f.: graviton

Possible issues of convergence: naively - Planck scale (?)

Ostrogradsky instability

Other (heavy) d.o.f. - ? (massive states of strings?)

“Emergent” behavior: phonon, graviton, photon (?)

Gauge-string duality: a VERY brief introduction
Ludwig Wittgenstein’s view of duality (1892; 1953)

(The analogy stolen from Shamit Kachru’s talk at Simons Foundation, New York, Feb 27, 2019)
From brane dynamics to AdS/CFT correspondence

Open strings picture: dynamics of $N_c$ coincident D3 branes at low energy is described by

$$\mathcal{N} = 4 \text{ supersymmetric } SU(N_c) \text{ YM theory in 4 dim } Z_{YM}[J]$$

Closed strings picture: dynamics of $N_c$ coincident D3 branes at low energy is described by

conjectured exact equivalence

type IIB superstring theory on $AdS_5 \times S^5$ background

$$Z_{IIB}[J]$$

Maldacena (1997); Gubser, Klebanov, Polyakov (1998); Witten (1998)
\[ \mathcal{N} = 4 \quad \text{supersymmetric YM theory} \]

- **Field content:**

\[
A_\mu \quad \Phi_I \quad \Psi^A_\alpha \quad \text{all in the adjoint of } SU(N)
\]

\[ I = 1 \ldots 6 \quad A = 1 \ldots 4 \]

- **Action:**

\[
S = \frac{1}{g_{YM}^2} \int d^4x \tr \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_I)^2 - \frac{1}{2} [\Phi_I, \Phi_J]^2 \\
+ i \bar{\Psi} \Gamma^\mu D_\mu \Psi - \bar{\Psi} \Gamma^I [\Phi_I, \Psi] \right\}
\]

- **Large N: effective coupling = ‘t Hooft coupling**

\[ \lambda = g_{YM}^2 N \]

(super)conformal field theory = coupling doesn’t run
Gravitational fluctuations (and fluctuations of other fields)

\[ g^{(0)}_{\mu\nu} + h_{\mu\nu} \]
\[ A^{(0)}_\mu + a_\mu \]

"\[ \Box \]" \( h_{\mu\nu} = 0 \)  "\[ \Box \]" \( a_\mu = 0 \)

and B.C.

Quasinormal spectrum

\[ \omega = -iDq^2 + \cdots \]

\[ \partial_t j^0 + \partial_i j^i = 0 \]

\[ \partial_t j^0 = D \nabla^2 j^0 \]

Deviations from equilibrium

????
Quasinormal modes in "real life" and beyond

Quasinormal modes region

\[ c = -1 \]

\[ c = +1 \]
Spectral curves from holography

Dual black hole fluctuations reduce (in gauge-invariant variables) to ODEs such as

$$\Phi'' - \frac{(w^2 - q^2 f) f - zw^2 f'}{zf(w^2 - q^2 f)} \Phi' + \frac{w^2 - q^2 f}{zf^2} \Phi = 0$$

$$\Phi(z) = A \varphi_1(z) + B \varphi_2(z)$$

$$\Phi(z) = A z^{\Delta-} (1 + \cdots) + B z^{\Delta+} (1 + \cdots) \text{ for } z \rightarrow 0$$

I. Computing the retarded correlator: inc.wave b.c. at the horizon, normalized to 1 at the boundary

$$G^R \sim \frac{B}{A} + \text{contact terms}$$

II. Computing quasinormal spectrum: inc.wave b.c. at the horizon, Dirichlet at the boundary

$$\mathcal{A}(\omega, q) = 0$$ \text{ This is the full spectral curve.}
In quantum field theory, the dispersion relations such as

$$\omega = \pm v_s q - \frac{i}{2(\epsilon + P)} \left( \frac{4}{3} \eta + \zeta \right) q^2$$

appear as poles of the retarded correlation functions, e.g.

$$\langle T_{00}(k) T_{00}(-k) \rangle \sim \frac{q^2 T^4}{\omega^2 - q^2/3 + i\omega q^2/3\pi T}$$

- in the hydro approximation -  \[ \omega/T \ll 1, \quad q/T \ll 1 \]
Singularities of a (retarded) Green’s function in the complex frequency plane

Shear channel

\[ \omega = -iDq^2 + \cdots \]

Strong (infinite) coupling

Real spatial momentum \( q \)
Dispersion relations of the quasinormal modes for real q
Recall the condition for a critical point of a curve

\[ F(x_*, y_*) = 0, \ F_y(x_*, y_*) = 0, \ldots F^{(p)}_y(x_*, y_*) \neq 0 \]

This is actually a “level-crossing” condition

\[ F(x, y) = (y - y_*(x))^p(y - y_1(x)) \cdots (y - y_k(x)) = 0 \]

Thus for the quasinormal spectrum curve \( A(\omega, q) = 0 \) we expect \( \omega_1(q_*) = \omega_2(q_*) \)

Shear channel

\[ \omega = -iDq^2 + \cdots \]

This “quasinormal level crossing” can happen at complex \( q_* \)
Poles of the retarded energy-momentum tensor correlator in complex $\omega$-plane at complex $q^2 = |q^2|e^{i\theta}$, $\theta \in [0, 2\pi]$

Shear channel
Quasinormal modes level-crossing at complex q
Radii of convergence of hydrodynamic modes in N=4 SYM  
(at infinite ‘t Hooft coupling, from dual gravity)

Shear mode: \( w = w(q) = -i \frac{\eta}{\epsilon + P} q^2 + \cdots \) \(|q_*| \approx 1.49131 \times (2\pi T)\)

Sound mode: \( w = w_\pm(q) = \pm v_s q - i \frac{\zeta + \frac{4}{3} \eta}{\epsilon + P} q^2 + \cdots \) \(|q_*| = \sqrt{2} \times (2\pi T)\)

\( w = \frac{\omega}{2\pi T}, \ q = \frac{q}{2\pi T}; \eta, \zeta \) – shear & bulk viscosities; \( v_s \) – speed of sound; \( \epsilon, P \) – energy density & pressure

What about finite ‘t Hooft coupling?

Crude estimate: \(|q_{\text{sound}}^c| = \sqrt{3} \left( 1 - 15\zeta(3)\lambda^{-3/2} + \cdots \right)\)

Thus it appears that the radius of convergence is smaller at weaker coupling – hydrodynamics in strongly interacting systems is more “robust”? 
Interpolating between weak and strong coupling
Interpolation between weak and strong coupling: exact results are very rare (even at $T=0$)...

Example (old & beautiful): expectation value of a circular Wilson loop in $\mathcal{N} = 4 \, SU(N_c) \, SYM$ in $d = 4$ in the limit $N_c \to \infty$, $\lambda \equiv g_{YM}^2 N_C$

$$\langle W_C \rangle = \frac{2}{\sqrt{\lambda}} \, I_1 \left(2\sqrt{\lambda}\right)$$

$$\langle W_C \rangle = 1 + \frac{\lambda}{4} + \frac{\lambda^2}{48} + \cdots \quad \lambda \ll 1$$

$$\langle W_C \rangle \sim \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{2\lambda}}}{(2\lambda)^{3/4}} + \cdots \quad \lambda \gg 1$$

Erickson, Semenoff and Zarembo (2000)
Quantum field theories at finite temperature/density

Equilibrium
- entropy
- equation of state
- ........

Non-perturbative
- perturbative
- non-perturbative
- pQCD
- Lattice

Near-equilibrium
- transport coefficients
- emission rates
- ........

Perturbative
- perturbative
- non-perturbative
- kinetic theory
- Holography?
Pressure in perturbative QCD

![Graph showing the pressure in perturbative QCD as a function of $T/\Lambda_{\text{MS}}$ for different values of $g^a$ and $g^6(\ln(1/g)+0.7)$ with 4d lattice data points.](image-url)
Entropy density of $\mathcal{N} = 4$ SYM in the planar limit ($\mathcal{N} \to \infty$)

$$\frac{S}{S_0} \quad \mathcal{N} = 4 \text{ super-Yang-Mills}$$

![Graph showing the entropy density of super-Yang-Mills]

- **Weak-coupling to order $\lambda^{3/2}$**
- **Strong-coupling to order $\lambda^{-3/2}$**

Fig from Blaizot, Iancu, Kraemmer and Rebhan, hep-ph/0611393

For $\lambda \ll 1$

$$s/s_0 = 1 - \frac{3}{2\pi^2} \lambda + \frac{\sqrt{2} + 3}{\pi^3} \lambda^{3/2} + \cdots$$

Fotopoulos and Taylor, hep-th/9811224

For $\lambda \gg 1$

$$s/s_0 = \frac{3}{4} + \frac{45}{32} \zeta(3) \lambda^{-3/2} + \cdots$$

Gubser, Klebanov and Tseytlin, hep-th/9805156

$$s_0 = \frac{2\pi^2}{3} N_c^2 T^3$$ - Stefan-Boltzmann (free gas)
From weak to strong coupling in finite-temperature QFT:

WEAK COUPLING (PERTURBATIVE REGIME)
Hydrodynamic regime in kinetic theory

Hierarchy of times (e.g. in Bogolyubov picture of kinetic theory)

\[ \tau_{mft} \ll t \ll T_{\text{existence}} \]

- Mechanical description
- Kinetic theory
- Hydrodynamic approximation
- Equilibrium thermodynamics

Hierarchy of scales

\[ l_{mfp} \ll l \ll L \]

(L is a macroscopic size of a system)
The hydrodynamic regime (continued)

Degrees of freedom

Hydro regime: \( \tau_{\text{micro}} \ll \tau \ll \tau_{\text{global}} \) \( l_{\text{micro}} \ll l \ll L_{\text{global}} \)
Relaxation time in kinetic theory

Kinetic equation
\[
\frac{\partial F}{\partial t} + \frac{p_i}{m} \frac{\partial F}{\partial r^i} - \frac{\partial U(r)}{\partial r^i} \frac{\partial F}{\partial p_i} = C[F]
\]

Linearized by
\[
F(t, r, p) = F_0(r, p) [1 + \varphi(t, r, p)]
\]

Leads to
\[
\frac{\partial \varphi}{\partial t} - \frac{p_i}{m} \frac{\partial \varphi}{\partial r^i} + \frac{\partial U(r)}{\partial r^i} \frac{\partial \varphi}{\partial p_i} + L_0[\varphi]
\]

For spatially homogeneous distributions:
\[
\varphi(t, p) = e^{-\nu t} h(p)
\]

Eigenvalue problem:
\[
-\nu h = L_0[h]
\]

Solution:
\[
\varphi(t, p) = \sum_n C_n e^{-\nu_n t} h_n(p)
\]
Spectrum of linearized kinetic operator
(at zero spatial momentum, i.e. all hydro modes are at 0)

Wang Chang & Uhlenbeck (1952), Grad (1963)

a) Discrete spectrum, \( U = \alpha/r^4 \)

b) Continuous spectrum with a gap, \( U = \alpha/r^n, \ n > 4 \)

c) Continuous gapless spectrum, \( U = \alpha/r^n, \ n < 4 \)

d) Hod spectrum
Relaxation time in kinetic theory (continued)

\[ \varphi(t, p) = \sum_n C_n e^{-\nu_n t} h_n(p) \]

\[-\nu h = L_0[h] \]

The hierarchy of relaxation times is determined by the spectrum of the linearized kinetic operator

\[ \tau_R = 1/\nu_{\text{min}} \]

For weakly inhomogeneous systems:

\[ \frac{\partial F}{\partial t} + \frac{p_i}{m} \frac{\partial F}{\partial r^i} - \frac{\partial U(r)}{\partial r^i} \frac{\partial F}{\partial p_i} = -\frac{F - F_0}{\tau_R} \]

Krook-Gross-Bhatnagar (KGB) equation (1959) a.k.a. “RTA”

Transport is then essentially determined by the relaxation time, e.g. shear viscosity is

\[ \eta = \tau_R s T \]
Of course, the situation is significantly more complicated for generic weakly interacting quantum systems (relativistic or not) at finite temperature and/or density.

Resummations typically lead to effective kinetic theory (AGD, Popov, AMY++). Transport is determined by the spectrum of kinetic operator. Partial results exist, yet e.g. the analytic structure of correlators of gauge-invariant operators is generically unknown (but see recent work by Guy D. Moore, 1803.0073).


From weak to strong coupling in finite-temperature QFT:

STRONG COUPLING (NON-PERTURBATIVE REGIME)
Shear viscosity in $\mathcal{N} = 4$ SYM

perturbative thermal gauge theory

$\eta \sim \frac{1}{\lambda^2 \log \frac{1}{\lambda}}$

$\frac{1}{4\pi} + \frac{15\zeta(3)}{4\pi} \frac{1}{\lambda^{3/2}} + \cdots$

Correction to $\frac{1}{4\pi}$ Buchel, Liu, A.O.S., hep-th/0406264

Buchel, 0805.2683 [hep-th]; Myers, Paulos, Sinha, 0806.2156 [hep-th]
Viscosity-entropy ratio in Unitary Fermi gas

\[(\eta/s)_{\text{min}} \sim 1.38 \text{ in units of } \frac{\hbar}{4\pi k_B}\]

Coupling constant corrections to N=4 SYM transport coefficients

\[ S = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} \left( R + \frac{12}{L^2} + \gamma \mathcal{W} \right) \]

\[ \gamma = \lambda^{-3/2} \zeta(3)/8 \]

\[ \eta = \frac{\pi}{8} N_c^2 T^3 \left( 1 + 135 \gamma + \ldots \right) \]

\[ \tau_\Pi = \frac{(2 - \ln 2)}{2\pi T} + \frac{375 \gamma}{4\pi T} + \ldots \]

\[ \kappa = \frac{N_c^2 T^2}{8} \left( 1 - 10 \gamma + \ldots \right) \]

\[ \lambda_1 = \frac{N_c^2 T^2}{16} \left( 1 + 350 \gamma + \ldots \right) \]

\[ \lambda_2 = -\frac{N_c^2 T^2}{16} \left( 2 \ln 2 + 5 \left( 97 + 54 \ln 2 \right) \gamma + \ldots \right) \]

\[ \lambda_3 = \frac{25N_c^2 T^2}{2} \gamma + \ldots \]

Note: \[ 2\eta \tau_\Pi - 4\lambda_1 - \lambda_2 = 0 \]
Curvature squared corrections to transport coefficients of a (hypothetical) strongly coupled liquid

\[ S_{R^2} = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} \left[ R - 2\Lambda + L^2 \left( \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \right] \]

\[ \eta = \frac{r_+^3}{2\kappa_5^2} \left( 1 - 8 \left( 5\alpha_1 + \alpha_2 \right) \right) + \ldots \]

\[ \eta\tau_\Pi = \frac{r_+^2 \left( 2 - \ln 2 \right)}{4\kappa_5^2} \left( 1 - \frac{26}{3} \left( 5\alpha_1 + \alpha_2 \right) \right) - \frac{r_+^2 \left( 23 + 5 \ln 2 \right)}{12\kappa_5^2} \alpha_3 + \ldots \]

\[ \kappa = \frac{r_+^2}{2\kappa_5^2} \left( 1 - \frac{26}{3} \left( 5\alpha_1 + \alpha_2 \right) \right) - \frac{25r_+^2}{6\kappa_5^2} \alpha_3 + \ldots \]

\[ \lambda_1 = \frac{r_+^2}{4\kappa_5^2} \left( 1 - \frac{26}{3} \left( 5\alpha_1 + \alpha_2 \right) \right) - \frac{r_+^2}{12\kappa_5^2} \alpha_3 + \ldots \]

\[ \lambda_2 = -\frac{r_+^2 \ln 2}{2\kappa_5^2} \left( 1 - \frac{26}{3} \left( 5\alpha_1 + \alpha_2 \right) \right) - \frac{r_+^2 \left( 21 + 5 \ln 2 \right)}{6\kappa_5^2} \alpha_3 + \ldots \]

\[ \lambda_3 = -\frac{28r_+^2}{\kappa_5^2} \alpha_3 + \ldots \]

Note: \[ 2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0 \]
Non-perturbative Gauss-Bonnet corrections to transport coefficients of a (hypothetical) strongly coupled liquid

\[ S_{GB} = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} \left[ R + \frac{12}{L^2} + \frac{\lambda_{GB}}{2} L^2 \left( R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) \right] \]

\[ \eta = \frac{s}{4\pi} \gamma^2 = \frac{s}{4\pi} (1 - 4\lambda_{GB}) \]

\[ \tau_\Pi = \frac{1}{2\pi T} \left( \frac{1}{4} (1 + \gamma) \left( 5 + \gamma - \frac{2}{\gamma} \right) - \frac{1}{2} \log \left[ \frac{2(1+\gamma)}{\gamma} \right] \right) \]

\[ \lambda_1 = \frac{\eta}{2\pi T} \left( \frac{(1 + \gamma) \left( 3 - 4\gamma + 2\gamma^3 \right)}{2\gamma^2} \right) \]

\[ \lambda_2 = -\frac{\eta}{\pi T} \left( -\frac{1}{4} (1 + \gamma) \left( 1 + \gamma - \frac{2}{\gamma} \right) + \frac{1}{2} \log \left[ \frac{2(1+\gamma)}{\gamma} \right] \right) \]

\[ \lambda_3 = -\frac{\eta}{\pi T} \left( \frac{(1 + \gamma) \left( 3 + \gamma - 4\gamma^2 \right)}{\gamma^2} \right) \]

\[ \kappa = \frac{\eta}{\pi T} \left( \frac{(1 + \gamma) \left( 2\gamma^2 - 1 \right)}{2\gamma^2} \right) \]

\[ H(\lambda_{GB}) = 2\eta \tau_{\Pi} - 4\lambda_1 - \lambda_2 = -\frac{\eta}{\pi T} \frac{(1 - \gamma_{GB}) (1 - \gamma_{GB}^2) (3 + 2\gamma_{GB})}{\gamma_{GB}^2} = -\frac{40\lambda_{GB}^2 \eta}{\pi T} + \ldots \]

Brigante et al, 2008

Banerjee and Dutta, 2011

Grozdanov and AOS, 2014

Grozdanov and AOS, 2014

Grozdanov and AOS, 2014

Banerjee and Dutta, 2011
Cuts versus poles: a mystery

We should be able to interpolate between the two limits...
Singularities of stress-energy tensor Green’s function at infinite (black dots) and finite (black crosses and diamonds) coupling.

Earlier work: Stricker, 1307.2736 [hep-th]; Waeber, Schäfer, Vuorinen and Yaffe, 1509.02983 [hep-th].
Singularities of stress-energy tensor Green’s function in different regimes of viscosity-entropy ratio (shear channel)

\[
\frac{\eta}{s} \geq \frac{1}{4\pi}
\]

White squares: poles at infinite coupling
Crosses: poles at finite coupling
On the “unreasonable effectiveness” of kinetic theory at strong coupling

Recall that in kinetic theory \( \eta = \text{const} \ s \ \tau_R \ T \)

What happens at large but finite coupling, with \( \tau_R = 1/|\text{Im} \ \omega_F| \)?
“Applicability of hydrodynamics” as a function of coupling

Hydro OK

Hydro not applicable

"strong coupling"

"weak coupling"
POLES VS CUTS: FROM INFINITE TO ZERO COUPLING

\[ \text{Im}[\nu] \quad \text{Re}[\nu] \]

\[ \text{Re}[\nu] \quad \text{Im}[\nu] \]

\[ -12 \quad -8 \quad -4 \quad 0 \quad 4 \quad 8 \quad 12 \]

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\[ -12 \quad -8 \quad -4 \quad 0 \quad 4 \quad 8 \quad 12 \]
Transport peak of spectral functions at large finite coupling

\[ \eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt \, d^3 x \, e^{i\omega t} \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle \]

\[ \chi_{xy,xy}(k) = \int d^4 x \, e^{-ikx} \, \langle [T_{xy}(x)T_{xy}(0)] \rangle = -2 \text{Im} \, G^R_{xy,xy} \]

Viscosity is determined by the height of the peak of the spectral function at \( w=0 \).

The peak is affected by the singularities of the correlator in the complex \( w \) plane.

What kind of singularities? Are they the same at weak and strong coupling?
Transport peak in QCD at finite temperature (sketch)
Transport peak of spectral functions at large finite coupling

$\mathcal{N} = 4$ SYM

Gauss-Bonnet

Note: Black solid line is the spectral function at infinite coupling

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int \! dt \, d^3 x e^{i\omega t} \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle$$
Conclusions & open questions

WARNING: STATEMENTS IN RED NEED MATHEMATICAL JUSTIFICATION
THEY MAY TURN OUT TO BE BE ARTIFACTS!

Gauge-string duality is an excellent theoretical laboratory
for exploring strongly interacting quantum systems at finite temperature and density, in and out of equilibrium

We can explore some generic features of effective theories (hydro)

We observe breakdown of hydrodynamics at coupling-dependent value of a wave-vector. The dependence on coupling suggests that hydrodynamics has a wider applicability range at stronger coupling.

Our results suggest that kinetic theory results may be formally still applicable in the intermediate and strong coupling regime where the use of kinetic theory itself cannot be justified. In particular, transport peak is visible at large finite coupling due to inflow of poles. Compare to pQFT?

We observe qualitatively different analytic structure of correlators depending on whether

\[ \frac{\eta}{s} > \frac{1}{4\pi} \text{ or } \frac{\eta}{s} < \frac{1}{4\pi} \]

We observe level-crossing of quasinormal modes. Wider applications?
THANK YOU!